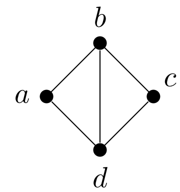
Spectral Properties Of Graphs

Adjacency matrix of a graph G of order n is $n \times n$ matrix A where

$$A_{ik} = \begin{cases} 1 & \text{if } ij \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

1: Find the adjacency matrix A for the following graph



Solution:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

2: Show that $A^k_{i,j}$ counts the number of walks from i to j of length k. If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k.

Solution: Induction on the length of path. To count the paths of length k from v_i to v_j, denote by $p^k_{i,j}$, we sum over all ℓ and counts paths from v_i to v_ℓ of length $k-1$ and paths from v_ℓ to v_j of length 1. Note that $p^1_{i,j} = A_{i,j}$. Goal is to show $p^k_{i,j} = A^k_{i,j}$.

$$p^k_{i,j} = \sum_\ell p^{k-1}_{i,\ell} p^1_{i,\ell} = \sum_\ell A^{k-1}_{i,\ell} A^1_{i,\ell} = A^k_{i,j}.$$

The **eigenvalues** of a graph are eigenvalues of its adjacency matrix. Since A is symmetric real, all eigenvalues are real numbers.

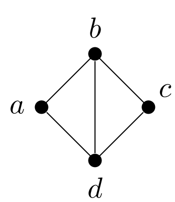
The eigenvalues are usually denoted as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. They are roots of the characteristic polynomial

$$det(\lambda I - A) = \prod_{i=1}^{n} (\lambda - \lambda_i)$$

Note The polynomial comes from $Av = \lambda v$, where v is an eigenvector.

Spectrum is the list of distinct eigenvalues λ_i with their multiplicities m_i, we denote as $Spec(G) = \begin{pmatrix} \lambda_1 \cdots \lambda_t \\ m_1 \cdots m_t \end{pmatrix}$.

3: Find the spectrum of



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Solution: $$\begin{vmatrix} \lambda & -1 & 0 & -1 \\ -1 & \lambda & -1 & -1 \\ 0 & -1 & \lambda & -1 \\ -1 & -1 & -1 & \lambda \end{vmatrix} = \begin{vmatrix} 0 & -1-\lambda & -\lambda & -1+\lambda^2 \\ 0 & \lambda+1 & 0 & -1-\lambda \\ 0 & -1 & \lambda & -1 \\ -1 & -1 & -1 & \lambda \end{vmatrix} = \begin{vmatrix} -1-\lambda & -\lambda & -1+\lambda^2 \\ \lambda+1 & 0 & -1-\lambda \\ -1 & \lambda & -1 \end{vmatrix}$$

$\lambda^4 - 5\lambda^2 - 4\lambda$, $Spec(G) = \begin{pmatrix} \frac{1+\sqrt{17}}{2} & 0 & \frac{1-\sqrt{17}}{2} & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ I just used Wolframalpha to calculate it `https://www.wolframalpha.com/input/?i=determinant%28%28x%2C-1%2C0%2C-1%29%2C%28-1%2Cx+%2C+-1+%2C+-1%29%2C%280%2C-1%2Cx%2C-1%29%2C%28-1%2C-1%2C+-1%2Cx%29%29`

4: Show that $\sum_{i=1}^{n} \lambda_i = Trace(A)$.

Solution: $Trace(A) = \sum_{i=1}^{n} A_{i,i}$. It is the negative coefficient of λ^{n-1} in $det(\lambda I - A)$. Since $det(\lambda I - A) = \prod_{i=1}^{n}(\lambda - \lambda_i)$, it equals to $\sum \lambda_i$. Note $Trace(A) = 0$ for adjacency matrix.

5: Show that adding c to the diagonal of A shifts the eigenvalues by c.

Solution: a is a root of $\det(\lambda I - A)$ if and only if $a + c$ is a root of $\det(\lambda I - (cI + A))$.

6: Determine the spectrum of K_n.

Solution: Once $n-1$ and the rest is -1. One can calculate the spectrum of J, the full 1 matrix and subtract 1 from the diagonal.
$Spec(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$

Lemma If G is a bipartite graph and λ is an eigenvalue of G with multiplicity m, then $-\lambda$ is also an eigenvalue of G with multiplicity m.

Proof. Obvious is $\lambda = 0$, so assume $\lambda \neq 0$.

7: Start by showing that we can assume G has both parts of the same order.

Solution: Add isolated vertices to the side with fewer vertices. Notice that adding an isolated vertex adds one 0 eigenvalue and does not change the others. Just see that A has one more row/column full of zeros.

Now we can assume that A has the form $\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$. If λ is an eigenvalue with eigenvector $v = \begin{pmatrix} x \\ y \end{pmatrix}$, we get

$$\lambda v = Av = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} By \\ B^T x \end{pmatrix}$$

Hence $By = \lambda x$ and $B^T x = \lambda y$.

8: Consider $v' = \begin{pmatrix} x \\ -y \end{pmatrix}$ and check that it is an eigenvector for $-\lambda$.

Solution:

$$Av' = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} B(-y) \\ B^T(x) \end{pmatrix} = \begin{pmatrix} \lambda(-x) \\ \lambda y \end{pmatrix} = -\lambda v'$$

Notice this show also the multiplicity since each eigenvector for λ gives an eigenvector for $-\lambda$.

9: Calculate the spectrum of $K_{m,n}$. What is the rank of A?

Solution: Rank of A is 2. Hence it has 2 non-zero eigenvalues λ_1 and λ_2. Since $\lambda_1 + \lambda_2 = Trace(A) = 0$, they differ just by the sign. Hence the characteristic polynomial is

$$det(\lambda I - A) = \prod_{i=1}^{n} (\lambda - \lambda_i) = \lambda^{n-2}(\lambda - \lambda_1)(\lambda + \lambda_1) = \lambda^n - \lambda_1^2 \lambda^{n-2}.$$

We need to get contribution to λ^{n-2} from $det(\lambda I - A)$. It means we pick $n-2$ entries from the diagonal and two off-diagonal entries as we calculate the determinant. Recall

$$det(B) = \sum_{\sigma \in S_n} \left(sgn(\sigma) \prod_{i=1}^{n} B_{i,\sigma_i} \right)$$

Here the permutations we pick are all odd, and entires are either 0 if i, j not edge or -1 if it is an edge. Hence $\lambda_1^2 = mn = |E(K_{m,n})|$.

$Spec(K_{m,n}) = \begin{pmatrix} \sqrt{mn} & 0 & -\sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}$

Lemma If G' is an induced subgraph of G, then

$$\lambda_{min}(G) \leq \lambda_{min}(G') \leq \lambda_{max}(G') \leq \lambda_{max}(G),$$

where λ_{min} and λ_{max} are the minimum, respectively maximum eigenvalues.

10: Prove the lemma. Hint: Use $\lambda_{min}(G) \leq x^T AX \leq \lambda_{max}(G)$ for every unit vector x since A is real and symmetric.

Solution: The idea is to take the an eigenvector of A' corresponding to $\lambda_{max}(G')$ and turn it into a unit vector of A, showing $\lambda_{max}(G') \leq \lambda_{max}(G)$. Min version is analogous.

Let G be an n vertex graph G on vertices $v_1, \ldots, v_n$. By relabeling, we assume G' is an induced subgraph of G on vertices $v_1, \ldots, v_k$. Let A, A' be the adjacency matrix of G, G' respectively. Notice that A is $n \times n$ matrix and A' is $k \times k$ submatrix of A sitting in the upper left corner. Let x' be a unit eigenvector vector of A' corresponding to the eigenvalue $\lambda_{max}(G')$. Let $x = v_1, \ldots, v_k, 0, \ldots 0$ have n entires. Notice x is also a unit vector. Hence

$$\lambda_{max}(G') = (x')^T A' x' = x^T A x \leq \lambda_{max}(G).$$

Interlacing theorem Let G be a graph on n vertices with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let G' be obtained from G by removing one vertex. Let the eigenvalues of G' be $\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{n-1}$. Then

$$\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \cdots \lambda'_{n-1} \geq \lambda_n.$$

11: Show that for any graph G

$$\lambda_{max}(G) \leq \Delta(G).$$

Hint: Consider eigenvector x corresponding to $\lambda_{max}(G)$ and its largest entry.

Solution: Let λ be an eigenvalue of the adjacency matrix A of a graph G. Let x be an eigenvector corresponding λ. Let x_j be the largest entry in x. onsider what happens to it when x is multiplied by A.

$$\lambda_a x_j = (Ax)_j = \sum_{v_i \in N(v_j)} x_i \leq d(v_j)x_j \leq \Delta(G)x_j$$

Hence $\lambda \leq \Delta(G)$ for all eigenvalues of A.

12: Show that for any graph G on n vertices and m edges

$$\delta(G) \leq \frac{2m}{n} \leq \lambda_{max}(G).$$

Hint: Consider unit vector with all coordinates $\frac{1}{\sqrt{n}}$ and use $\lambda_{max} \geq x^T Ax$ over all unit vectors.

Solution: Let x be a unit vector with coordinates $\frac{1}{\sqrt{n}}$. Notice that the sum of all entries in the adjacency matrix is the sum of degrees, which is $2m$.

$$\lambda_{max} \leq x^x Ax = \frac{1}{n}\sum_{i,j} a_{i,j} = \frac{2m}{n} \geq \delta(G).$$

Note that $\frac{2m}{n}$ is the average degree, hence it is at least the minimum degree.

Theorem (Wilf 1967) $\chi(G) \leq \lambda_{max}(G)$

13: Prove the theorem. Use that if $\chi(G) = k$, then G has a subgraph with minimum degree $k-1$.

Solution: If $\chi(G) = k$, then G has an induced subgraph H of minimum degree $k-1$. It can be obtained by iteratively removing vertices of degree $k-2$. Since it is an induced subgraph, we have

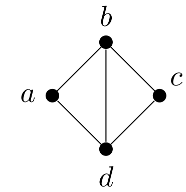
$$k \leq 1 + \delta(H) \leq 1 + \lambda_{max}(H) \leq 1 + \lambda_{max}(G).$$

Laplace matrix of a graph G of order n is $n \times n$ matrix Q where

$$q_{ij} = \begin{cases} -1 & \text{if } ij \in E(G) \\ deg(i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Eigenvalues of the Laplacian Matrix are sometimes denoted by $0 = \mu_1 \leq \ldots \leq \mu_n$. Note they are ordered in the opposite direction than eigenvalues of the adjacency matrix.

14: Find the spectrum and characteristic polynomial of the Laplacian matrix for

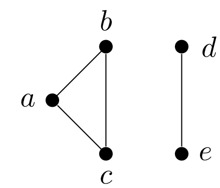


Solution:

$$\begin{vmatrix} \lambda-2 & 1 & 0 & 1 \\ 1 & \lambda-3 & 1 & 1 \\ 0 & 1 & \lambda-2 & 1 \\ 1 & 1 & 1 & \lambda-3 \end{vmatrix} = \lambda^4 - 10\lambda^3 + 32\lambda^2 - 32\lambda = (\lambda-4)^2(\lambda-2)\lambda$$

The spectrum is $0, 2, 4, 4$.

15: Find the spectrum and characteristic polynomial of the Laplacian matrix for



Solution:

$$\begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Eigenvalues: 0,0,2,3,3
Characteristic polynomial $18\lambda^2 - 21\lambda^3 + 8\lambda^4 - \lambda^5$

16: What is the coefficient of λ^{n-1} of the characteristic polynomial the Laplacian matrix?

Solution: When calculating It sums the diagonal entries, which is the sum of the degrees with a negative sign. Notice that when calculating the determinant of $(\lambda I - L)$, the coefficient for λ^{n-1} must come from the permutation that takes only diagonal entries and exactly one of them picks term $-deg(i)$ and all others it picks λ. This sums over all i.

17: Why is 0 always an eigenvalue of the Laplacian matrix?

Solution: Because all 1 vector is always an eigenvector.

18: What is the coefficient of λ of the characteristic polynomial the Laplacian matrix?

Solution: Product of the non-zero eigenvalues.

19: Show the multiplicty 0 as an eigenvalue of a Laplacian matrix of G is at least the number of connected components of G.

Solution: If G is not connected, then the characteristic vector for each component is an eigenvector with 0 eigenvalue. Hence the multiplicity of 0 is at least the number of connected components.

Theorem Kelmans (1967) The number of spanning trees in a graph whose Laplacian eigenvalues are $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ is

$$\frac{1}{n}\prod_{i=2}^{n}\mu_i.$$

Solution: Recall determinant of a matrix is a product of it's eigenvalues.

Theorem Fiedler (1973) For a non-complete graph G, connectivity is bounded from below by algebraic connectivity. That is $\kappa(G) \geq \mu_2(G)$.

More resources

Steve's MAA lecture
https://www.youtube.com/watch?v=ISUugS3mpL8

Steve's Spectral Graph Theory class
https://www.youtube.com/watch?v=Ft3xygCaP7c&list=PLi4h0n4UP8d9VGPqO8vLQga9ZApO65TLW